Submanifolds of complex space forms and twistor space of complex 2-plane Grassmannian

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September 9, 2016

Quaternionic Differential Geometry and related topics
Ochanomizu University

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- Hopf hypersurfaces in $\mathbb{CH}^n$ and para-quaternionic Kähler geometry of $\mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$,
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Gauss map of hypersurfaces in sphere

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- Then the Gauss map $\gamma : M \to \widetilde{G}_2(\mathbb{R}^{n+2}) \cong Q^n$ is defined by
- $\gamma(p) = x(p) \wedge N_p$ (B. Palmer, 1997).

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- Moreover, if $M^n \subset S^{n+1}$ is either isoparametric or austere, then $\gamma(M) \subset Q^n$ is a minimal Lagrangian submanifold.
- Also for parallel hypersurface $M_r := \cos rx + \sin rN$ $(r \in \mathbb{R})$ of M, the Gauss image is not changed: $\gamma(M) = \gamma(M_r)$.

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- Then we have a lift $\tilde{\gamma} : M^n \to V_2(\mathbb{R}^{n+2})$ to real Stiefel manifold, and with respect to a contact structure of $V_2(\mathbb{R}^{n+1})$, $\tilde{\gamma}$ is a Legendrian immersion.

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- Anciaux (2014) generalized the result to hypersurfaces in hyperbolic space and indefinite real space forms.

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- For $p \in M$, take a point $z_p \in \pi^{-1}(x(p)) \subset \pi^{-1}(M)$ and let N'_p be a holizontal lift of unit normal of $M \subset \mathbb{C}P^n$ at z_p.

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- If we put $\gamma(p) = \mathrm{span}_{\mathbb{C}}\{z_p, N'_p\}$,then the map $\gamma : M \to \mathbb{G}_2(\mathbb{C}^{n+1})$ is well-defined.

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- We call γ as the Gauss map of real hypersurface M in $\mathbb{CP}^n$.
- Note that for parallel hypersurface $M_r := \pi(\cos r z_p + \sin r N'_p)$ of M,image of the Gauss map $\gamma : M^{2n-1} \to \mathbb{CP}^n$ is not changed: $\gamma(M) = \gamma(M_r)$.

Hopf hypersurfaces in Kähler manifold

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- If $\widetilde{M}$ is a non-flat complex space form $\widetilde{M}(c)$ $(c \neq 0)$, then μ is constant on M (Y. Maeda and Ki-Suh) and
- each integral curve of ξ is a circle in $\mathbb{CP}^1 \subset \mathbb{CP}^n$ (resp. $\mathbb{CH}^1 \subset \mathbb{CH}^n$) when $c > 0$ (resp. $c < 0$).

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- Conversely, if a Hopf hypersurface M in $\mathbb{CP}^n(4)$ satisfies $A\xi = \mu\xi$, and for $r \in (0, \pi/2)$ with $\mu = 2\cot 2r$, $r \in (0, \pi/2)$, if rank of the focal map $\phi_r : M \to \mathbb{CP}^n$ is constant, then

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- $\phi_r(M)$ is a complex submanifold of $\mathbb{CP}^n(4)$ and M lies on a tube over $\phi_r(M)$. (Cecil-Ryan, 1982).

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- In this talk, we will give a characterization of Hopf hypersurface M in $\mathbb{CP}^n$ by using the Gauss map $\gamma : M \to \mathbb{G}_2(\mathbb{C}^{n+2})$.

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- We note that

Quaternionic Kähler manifold

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- Here, $\tilde{g}$ is a Riemannian metric of $\widetilde{M}$, Q is a subbundle of End$T\widetilde{M}$ with rank 3, satisfying:
- For each $p \in \widetilde{M}$, there exists a neighborhood $U \ni p$, such that there exists local frame field $\{\tilde{I}_1, \tilde{I}_2, \tilde{I}_3\}$ of Q.

Quaternionic Kähler manifold

- $$\tilde{I}_1^2 = \tilde{I}_2^2 = \tilde{I}_3^2 = -1, \quad \tilde{I}_1\tilde{I}_2 = -\tilde{I}_2\tilde{I}_1 = \tilde{I}_3,$$
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- For each $L \in Q_p$, $\tilde{g}$ is invariant, i.e., $\tilde{g}_p(LX, Y) + \tilde{g}_p(X, LY) = 0$ for $X, Y \in T_p\widetilde{M}$, $p \in \widetilde{M}$.

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- Vector bundle Q is parallel with respect to the Levi-Civita connection of $\tilde{g}$ at $\mathrm{End}\, T\widetilde{M}$.

Almost Hermitian submanifolds in Q.K. manifold

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- if we write the almost complex structure on M which is induced by $\tilde{I}$ as I, then
- with respect to the induced metric, (M, I) is an almost Hermitian manifold.

Totally complex submanifold of Q.K. manifold

- In particular, when almost Hermitian submanifold $(M, \bar{g}, I)$ is Kähler, we call M a Kähler submanifold of quaternionic Kähler manifold $\widetilde{M}$.

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- Similarly, an almost Hermitian submanifold $(M, \bar{g}, I)$ is called totally complex submanifold if at each point $p \in M$, with respect to $\tilde{L} \in Q_p$ which anti-commute with $\tilde{I}_p$, $\tilde{L}T_pM \perp T_pM$ hold.

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- In quaternionic Kähler manifold, a submanifold is totally complex if and only if it is Kähler (Alekseevsky-Marchiafava, 2001).

Gauss map of real hypersurface in $\mathbb{CP}^n$

- Theorem 1. (K., Diff. Geom. Appl. 2014) Let M^{2n-1} be a real hypersurface in complex projective space $\mathbb{CP}^n$, and let $\gamma : M \to \mathbb{G}_2(\mathbb{C}^{n+1})$ be the Gauss map.

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- If M is not Hopf, then the Gauss map γ is an immersion.
- If M is a Hopf hypersurface, then the image $\gamma(M)$ is a half-dimensional totally complex submanifold of $\mathbb{G}_2(\mathbb{C}^{n+1})$.
- And a Hopf hypersurface M in $\mathbb{CP}^n$ is a total space of a circle bundle over a Kähler manifold such that the fibration is nothing but the Gauss map $\gamma : M \to \gamma(M)$.

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- If $\widetilde{M}$ has positive Ricci curvature, then $\mathcal{Z}$ admits an Einstein-Kähler metric with positive Ricci curvture,
- such that the twistor fibration $\pi : \mathcal{Z} \to \widetilde{M}$ is a Riemannian submersion with totally geodesic fibers.

Twistor space of $G_2(\mathbb{C}^{n+1})$

- Recently K. Tsukada investigated twistor space $\mathcal{Z}$ of complex $\mathbf{2}$-plane Grassmannian $\mathbb{G}_{\mathbf{2}}(\mathbb{C}^{\boldsymbol{n+1}})$ (Diff. Geom. Appl. 2016),

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- and he showed that $\mathcal{Z}$ is identified with the projective cotangent bundle $P(T^*\mathbb{CP}^n)$ of a complex projective space $\mathbb{CP}^n$.
- As a homogeneous space, $\mathcal{Z}$ is expressed as $U(n+1)/U(n-1) \times U(1) \times U(1)$.

Q.K. structure on $\mathbb{G}_2(\mathbb{C}^{n+1})$

- Let $V_2(\mathbb{C}^{n+1})$ be the complex Stiefel manifold of orthonormal 2-vectors (u_1, u_2) in $\mathbb{C}^{n+1}$, and

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- let $\pi^G : V_2(\mathbb{C}^{n+1}) \to \mathbb{G}_2(\mathbb{C}^{n+1})$ be the projection defined by $(u_1, u_2) \mapsto \mathbb{C}u_1 \oplus \mathbb{C}u_2$.

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- Then tangent space $T_{\pi^G(u_1,u_2)}(\mathbb{G}_2(\mathbb{C}^{n+1}))$ is identified with $\{u_1, u_2\}^\perp \times \{u_1, u_2\}^\perp$ in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ through π^G_*.

Q.K. structure on $\mathbb{G}_2(\mathbb{C}^{n+1})$

- With respect to $(\boldsymbol{u_1}, \boldsymbol{u_2}) \in \boldsymbol{V_2}(\mathbb{C}^{\boldsymbol{n+1}})$, a basis $\boldsymbol{I_1}, \boldsymbol{I_2}$ and $\boldsymbol{I_3}$ of Q.K. structure of $\mathbb{G}_{\boldsymbol{2}}(\mathbb{C}^{\boldsymbol{n+1}})$ is given by: for $(\boldsymbol{x_1}, \boldsymbol{x_2}) \in \{\boldsymbol{u_1}, \boldsymbol{u_2}\}^{\perp} \times \{\boldsymbol{u_1}, \boldsymbol{u_2}\}^{\perp}$,

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- $I_1 : (x_1, x_2) \mapsto (x_1, x_2)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (x_2, -x_1)$,
- $I_2 : (x_1, x_2) \mapsto (x_1, x_2)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = (ix_1, -ix_2)$,

Q.K. structure on $\mathbb{G}_2(\mathbb{C}^{n+1})$

- With respect to $(u_1, u_2) \in V_2(\mathbb{C}^{n+1})$, a basis I_1, I_2 and I_3 of Q.K. structure of $\mathbb{G}_2(\mathbb{C}^{n+1})$ is given by: for $(x_1, x_2) \in \{u_1, u_2\}^\perp \times \{u_1, u_2\}^\perp$,
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- From this, we may identify the twistor space $\mathcal{Z}$ of $\mathbb{G}_2(\mathbb{C}^{n+1})$ and the space of concentric circles in $\mathbb{CP}^1 \subset \mathbb{CP}^n$, and
- also $\mathcal{Z}$ is identified with with the space of oriented geodesics in $\mathbb{CP}^1 \subset \mathbb{CP}^n$.

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- Also $\pi(\psi(M))$ is a totally complex submanifold of $\mathbb{G}_2(\mathbb{C}^{n+1})$.

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- Since Σ is a totally complex submanifold of $\mathbb{G}_2(\mathbb{C}^{n+1})$, $\tilde{I}(\Sigma)$ is a Legendrian submanifold of the twistor space $\mathcal{Z}$ with respect to a complex contact structure (Alekseevsky-Marchiafava, 2004).

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- $$\begin{array}{ccccc} \tilde{I}^* E & \xrightarrow[\eta]{} & E & \xrightarrow[\psi]{} & \mathbb{CP}^n \\ \downarrow & & \downarrow & & , \\ \Sigma^{n-1} & \xrightarrow[\tilde{I}]{} & \mathcal{Z} \cong L(\mathbb{CP}^n) & & \end{array}$$

Converse construction

- The map $\Phi := \psi \circ \eta : \tilde{I}^*E \to \mathbb{CP}^n$ gives Hopf hypersurface with $A\xi = 0$ (on open subset of regular points of $M = \tilde{I}^*E$), and

Converse construction

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- its parallel hypersurface $\phi_r(\tilde{I}^* E)$ gives Hopf hypersurface with $A\xi = 2\tan 2r\xi$ (on open subset of regular points of $M = \tilde{I}^* E$).

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Remarks

- Recently K. Tsukada proved that conormal bundle of any complex submanifold in $\mathbb{CP}^n$ is realized as a half dimensional totally complex submanifold in $\mathbb{G}_2(\mathbb{C}^{n+1})$.
- For real hypersurfaces in complex hyperbolic space $\mathbb{CH}^n$, we define Gauss map $\gamma : M \to \mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$, and
- we obtain similar results for Hopf hypersurfaces in $\mathbb{CH}^n$ by using para-quaternionic Kähler structure (J.T. Cho and M.K., Topol. Appl. 2015).

Complex hyperbolic space $\mathbb{C}\mathbb{H}^n$

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- $$\langle z, w \rangle = \mathrm{Re}\left(-z_0\bar{w}_0 + \sum_{k=1}^{n} z_k \bar{w}_k \right),$$

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$$H_1^{2n+1} = \{z \in \mathbb{C}_1^{n+1} \mid \langle z, z\rangle = -1\}.$$
- H_1^{2n+1} is the principal fiber bundle over $\mathbb{CH}^n$ with the structure group S^1 and the fibration $\pi : H_1^{2n+1} \to \mathbb{CH}^n$.

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- Then we have a Gauss map $G : M^{2n-1} \to G_{1,1}(\mathbb{C}_1^{n+1})$ of real hypersurface M in $\mathbb{CH}^n$.

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- provided that the rank of the focal map is constant as Cecil-Ryan's Theorem.

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- Structure theorem for Hopf hypersurfaces with $\mu = \pm 2$ was not known.

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- Each integral curve of ξ is a horocycle lies in $\mathbb{CH}^1 \subset \mathbb{CH}^n$, provided $|\mu| = 2$.

Split-quaternions

- $\widetilde{\mathbb{H}} = C(2,0) = C(1,1)$, Split-quaternions (or coquaternions, para-quaternions):
  $q = q_0 + iq_1 + jq_2 + kq_3$, $i^2 = -1$, $j^2 = k^2 = 1$,
  $ij = -ji = -k$, $jk = -kj = i$, $ki = -ik = -j$,
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- Introduced by James Cockle in 1849.

Para-quaternionic structure

- $\{I_1, I_2, I_3\}$, $I_1^2 = -1$, $I_2^2 = I_3^2 = 1$,
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- $\tilde{V} = \{aI_1 + bI_2 + cI_3 | \, a, b, c \in \mathbb{R}\} \cong \mathfrak{su}(1,1) \cong \mathbb{R}^3_1$, and
- $Q_+ = \{I \in \tilde{V} | \, I^2 = 1\} \cong S^2_1$: de-Sitter space, $Q_- = \{I \in \tilde{V} | \, I^2 = -1\} \cong H^2$: hyperbolic space, $Q_0 = \{I \in \tilde{V} | \, I^2 = 0, \; I \neq 0\} \cong$ lightcone.

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- For any element $L \in \tilde{Q}_p$, $\tilde{g}_p$ is invariant by L, i.e., $\tilde{g}_p(LX, Y) + \tilde{g}_p(X, LY) = 0$ for $X, Y \in T_p\widetilde{M}$, $p \in \widetilde{M}$.

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- The vector bundle $\tilde{Q}$ is parallel in $\mathrm{End}\, T\widetilde{M}$ with respect to the pseudo-Riemannian connection $\tilde{\nabla}$ associated with $\tilde{g}$.
- Complex $(1,1)$-plane Grassmannian $\mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$ is an example of para-quaternionic Kähler manifold.

- Let $V_{1,1}(\mathbb{C}_1^{n+1})$ be the complex Stiefel manifold of orthonormal timelike and spacelike vectors (u_-, u_+) in $\mathbb{C}_1^{n+1}$, and

Para-Q.K. structure on $\mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$

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- let $\boldsymbol{\pi}^{\boldsymbol{G}} : \boldsymbol{V_{1,1}}(\mathbb{C}_{\mathbf{1}}^{\boldsymbol{n+1}}) \to \mathbb{G}_{\mathbf{1,1}}(\mathbb{C}_{\mathbf{1}}^{\boldsymbol{n+1}})$ be the projection defined by $(\boldsymbol{u}_-, \boldsymbol{u}_+) \mapsto \mathbb{C}\boldsymbol{u}_- \oplus \mathbb{C}\boldsymbol{u}_+$.

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- Let $V_{1,1}(\mathbb{C}_1^{n+1})$ be the complex Stiefel manifold of orthonormal timelike and spacelike vectors (u_-, u_+) in $\mathbb{C}_1^{n+1}$, and
- let $\pi^G : V_{1,1}(\mathbb{C}_1^{n+1}) \to \mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$ be the projection defined by $(u_-, u_+) \mapsto \mathbb{C}u_- \oplus \mathbb{C}u_+$.
- Then tangent space $T_{\pi^G(u_-,u_+)}(\mathbb{G}_{1,1}(\mathbb{C}_1^{n+1}))$ is identified with $\{u_-, u_+\}^\perp \times \{u_-, u_+\}^\perp$ in $\mathbb{C}_1^{n+1} \times \mathbb{C}_1^{n+1}$ through π_*^G.

Para-Q.K. structure on $\mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$

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- Then we have $I_1^2 = -1$ and $I_2^2 = I_3^2 = 1$.

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- such that $dg_x(T_xM)$ is invariant under $\tilde{I}_{\tilde{1}}$ and $\tilde{I}_{\tilde{2}}dg_x(T_xM), \tilde{I}_3dg_x(T_xM)$ are orthogonal to $dg_x(T_xM)$.

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- satisfies $p + q \leq n - 1$.

Circles in $\mathbb{CH}^1$ in $\mathbb{CH}^n$

- Each fiber of the *twistor space* $\mathcal{Z}_-$ (resp. $\mathcal{Z}_+$ and $\mathcal{Z}_0$) satsfying $I^2 = -1$ (resp. $I^2 = 1$ and $I^2 = 0$) of $\mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$ is identified with hyperbolic plane H (resp. de Sitter plane S_1^2 and lightcone C) in a Lie algebra $\mathfrak{su}(1,1)$, and

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- for each corresponding 1-parameter subgroup $\exp(sX)$ $(X \in S)$, orbits in the complex hyperbolic line $[u_-, u_+] = \mathbb{CH}^1$ in $\mathbb{CH}^n$ are concentric circles (resp. equidistance curves of a geodesic and horocycles).

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- the space of concentric circles (resp. equidistance curves of a geodesic and horocycles) in $\mathbb{CH}^1 \subset \mathbb{CH}^n$.

Theorem 3

- Let M^{2n-1} be a Hopf hypersurface in $\mathbb{CH}^n$ with Hopf curvature μ with $|\mu| = 2$. For each point p in M, let $\psi(p)$ be the integral curve (horocycle) of ξ through p.

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- We denote $\psi^* E_0$ the pullback bundle of E_0 over Σ.

Theorem 3

- We have a map $\boldsymbol{\Phi_0} : \boldsymbol{\psi^* E_0} \to \mathbb{CH}^{\boldsymbol{n}}(\boldsymbol{-4})$ such that each fiber of $\boldsymbol{\psi^* E_0} \to \boldsymbol{\Sigma}$ is mapped to a horocycle in $\mathbb{CH}^{\boldsymbol{1}} \subset \mathbb{CH}^{\boldsymbol{n}}$.

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- Similar results hold for Hopf hypersurfaces M^{2n-1} in $\mathbb{CH}^n(-4)$ with Hopf curvature $\mu \neq \pm 2$ and horizontal submanifolds in the twistor spaces $\mathcal{Z}_\pm$.
- Hence any Hopf Hypersurfaces in $\mathbb{CH}^n$ is treated unified way.

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- (i) 'totally real' w.r.t. the standard complex structure of $\mathbb{G}_2(\mathbb{C}^{n+1})$ and (ii) there exists a section $\tilde{I}$ to $Q|_\Sigma$ such for each section I to $Q|_\Sigma$ which anticommutes with $\tilde{I}$, $I(T\Sigma) \perp T\Sigma$ holds.