Hopf hypersurfaces in non-flat complex space forms

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- the structure vector of Σ is defined by $\boldsymbol{\xi} = -JN$,
- where J is the complex structure of M.

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- the structure vector $\boldsymbol{\xi} = -JN$ is an eigenvector of the shape operator A of M, i.e., $A\boldsymbol{\xi} = \mu\boldsymbol{\xi}$.
- It is known that when $c \neq 0$, the principal curvature μ (Hopf curvature) of ξ on Hopf hypersurface M in $\widetilde{M}(c)$ is constant (Y.Maeda, Ki-Suh).

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- provided that the focal map $\phi_r: M \to \mathbb{CP}^n$, $\phi_r(x) = \exp_x(rN_x) \ (x \in M, N_x \in T_x^{\perp}M, |N_x| = 1)$ has constant rank on M.

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- provided that the rank of the focal map is constant as Cecil-Ryan's Theorem.

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- a Hopf hypersurface with $|\mu| < 2$ in $\mathbb{C}\mathbb{H}^n$ may be constructed from
- an arbitrary pair of Legendrian submanifolds in S^{2n-1} .

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- in terms of Gauss maps from M to some Grassmannians:
- $M^{2n-1}\hookrightarrow \mathbb{CP}^n \iff \Sigma^{2n-2} \hookrightarrow G_2(\mathbb{C}^{n+1})$,
- $M^{2n-1} \hookrightarrow \mathbb{C}\mathbb{H}^n \iff \Sigma^{2n-2} \hookrightarrow G_{1,1}(\mathbb{C}_1^{n+1}).$

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- Then we have a Gauss map $G: M^{2n-1} \to G_2(\mathbb{C}^{n+1})$ of real hypersurface M in \mathbb{CP}^n .

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- For each $p\in \widetilde{M}$, there is a neighborhood U of p over which there exists a local frame field $\{\tilde{I}_1,\tilde{I}_2,\tilde{I}_3\}$ of Q satisfying

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• For any element $L \in Q_p$, \tilde{g}_p is invariant by L, i.e., $\tilde{g}_p(LX, Y) + \tilde{g}_p(X, LY) = 0$ for $X, Y \in T_p\widetilde{M}$, $p \in \widetilde{M}$.

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- The vector bundle Q is parallel in End TM with respect to the Riemannian connection $\widetilde{\nabla}$ associated with \tilde{g} .
- Complex 2-plane Grassmannian $G_2(\mathbb{C}^{n+1})$ is an example of quaternionic Kähler manifold.

Almost Hermitian submanifolds in Q-K manifolds

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• We denote by I the almost complex structure on M induced from $ilde{I}$.

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- If (M, g, I) is Kähler, we call it a Kähler submanifold of \widetilde{M} .
- An almost Hermitian submanifold M together with a section \tilde{I} of $Q|_M$ is said to be totally complex if at each point $p \in M$, we have $LT_pM \perp T_pM$, for each $L \in Q_p$ with $\tilde{g}(L, \tilde{I}_p) = 0$.

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- It is known that a 2m $(m \ge 2)$ -dimensional almost Hermitian submanifold M^{2m} is Kähler if and only if it is totally complex.



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- let $G: M o G_2(\mathbb{C}^{n+1})$ be the Gauss map.
- Suppose M is a Hopf hypersurface.
- Then the image G(M) is a real (2n-2)-dimensional totally complex submanifold of $G_2(\mathbb{C}^{n+1})$.

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- We have another projection $\mathrm{pr}_1: E o \mathbb{CP}^n$, $\mathrm{pr}_1(\ell_1,\ell_2) = \ell_1$.

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- An almost complex structure \tilde{I} in Q_p , where $p \in G_2(\mathbb{C}^{n+1}) \cong \{\mathbb{CP}^1 \subset \mathbb{CP}^n\}.$

• Let $\varphi: \Sigma^{n-1} \to G_2(\mathbb{C}^{n+1})$ be a totally complex immersion, where Σ is a Kähler manifold, and

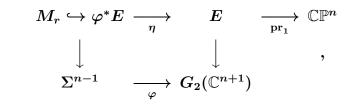
- Let $\varphi: \Sigma^{n-1} \to G_2(\mathbb{C}^{n+1})$ be a totally complex immersion, where Σ is a Kähler manifold, and
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- We denote M_r $(0 \le r < \pi/4)$ as a circle bundle over Σ such that each fiber of $M_r \to \Sigma$ is the equi-distance curve (circle) from the geodesic in $\mathbb{CP}^1 \cong \varphi(p)$ $(p \in \Sigma)$ corresponding to $\tilde{I} \in Q_{\varphi(p)}$.

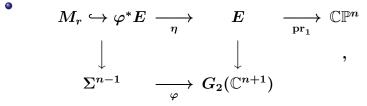
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- Hence $\Phi_r := \operatorname{pr}_1 \circ \eta : M_r \to \mathbb{CP}^n$ gives a real hypersurface provided that Φ_r is an immersion.

• For a totally complex immersion $\varphi: \Sigma^{n-1} \to G_2(\mathbb{C}^{n+1})$, we have a (local) frame field $g: \Sigma \to U(n+1)$, $g(p) = (u_1(p), u_2(p), \cdots, u_{n+1}(p))$ such that

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- We denote ω_{ij} $(1 \le i,j \le n+1)$ as components of the pull-back of Maurer-Cartan form $g^{-1}dg$ on U(n+1).



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- suppose the map $\Phi_r: M_r \to \mathbb{CP}^n$ is an immersion, where $M_r \to \Sigma$ is a circle bundle as above.
- Then $\Phi_r(M_r)$ is a Hopf hypersurface in \mathbb{CP}^n provided that $\operatorname{Im} \omega_{11} = \operatorname{Im} \omega_{22}$ and $\operatorname{Im} \omega_{12} = 0$ hold.



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• H_1^{2n+1} is the principal fiber bundle over \mathbb{CH}^n with the structure group S^1 and the fibration $\pi: H_1^{2n+1} \to \mathbb{CH}^n$.

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 with $x\in V.$

- Let $(V = \mathbb{R}(p, q), \langle , \rangle)$ be a real symmetric inner product space of signature p, q.
- The Clifford algebra C(p,q) is the quotient $\otimes V/I(V)$, where I(V) is the two-sided ideal in $\otimes V$ generated by all elements:
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$$x\otimes x+\langle x,x
angle$$
 with $x\in V.$

• Clifford algebras C(p,q) with small p,q are:

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• *J*, *J*² = 1 gives structures: almost product, para-complex, para-Hermitian, para-Kähler.

Quaternions and quaternionic structure

• $\mathbb{H} = C(0,2)$, Quaternions: $q = q_0 + iq_1 + jq_2 + kq_3$, $i^2 = j^2 = k^2 = ijk = -1$, $|q|^2 = q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$, no zero divisors, \rightsquigarrow

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- $V = \{aI_1 + bI_2 + cI_3 | a, b, c \in \mathbb{R}\} \cong \mathfrak{su}(2) \cong \mathbb{R}^3$, and $Q = \{I \in V | I^2 = -1\} \cong S^2$.

Split-quaternions

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$$\widetilde{\mathbb{H}} = C(2,0) = C(1,1)$$
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coquaternions, para-quaternions):
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- http://en.wikipedia.org/wiki/Split-quaternion
- Introduced by James Cockle in 1849.

Para-quaternionic structure

• $\{I_1, I_2, I_3\}$, $I_1^2 = -1$, $I_2^2 = I_3^2 = 1$, $I_1I_2 = -I_2I_1 = -I_3$, $I_2I_3 = -I_3I_2 = I_1$, $I_3I_1 = -I_1I_3 = -I_2$ gives para-quaternionic structure,

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- $ilde{V}=\{aI_1+bI_2+cI_3|~a,b,c\in\mathbb{R}\}\cong\mathfrak{su}(1,1)\cong\mathbb{R}^3_1,$ and
- $Q_+ = \{I \in \tilde{V} | I^2 = 1\} \cong S_1^2$: de-Sitter space, $Q_- = \{I \in \tilde{V} | I^2 = -1\} \cong H^2$: hyperbolic space, $Q_0 = \{I \in \tilde{V} | I^2 = 0\} \cong$ lightcone.

• Let $(\widetilde{M}^{4m}, \tilde{g}, \tilde{Q})$ be a para-quaternionic Kähler manifold with the quaternionic Kähler structure (\tilde{g}, \tilde{Q}) ,

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 $egin{array}{lll} ilde{I}_1^2 = -1, \; ilde{I}_2^2 = ilde{I}_3^2 = 1, \;\;\; ilde{I}_1 ilde{I}_2 = - ilde{I}_2 ilde{I}_1 = - ilde{I}_3, \;\;\; \ ilde{I}_2 ilde{I}_3 = - ilde{I}_3 ilde{I}_2 = ilde{I}_1, \;\;\; ilde{I}_3 ilde{I}_1 = - ilde{I}_1 ilde{I}_3 = - ilde{I}_2. \end{split}$

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• For any element $L \in \tilde{Q}_p$, \tilde{g}_p is invariant by L, i.e., $\tilde{g}_p(LX, Y) + \tilde{g}_p(X, LY) = 0$ for $X, Y \in T_p\widetilde{M}$, $p \in \widetilde{M}$.

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- The vector bundle \tilde{Q} is parallel in $\operatorname{End} TM$ with respect to the pseudo-Riemannian connection $\widetilde{\nabla}$ associated with \tilde{g} .
- Complex (1, 1)-plane Grassmannian $G_{1,1}(\mathbb{C}_1^{n+1})$ is an example of para-quaternionic Kähler manifold.

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- We denote by I the almost (para-)complex structure on M induced from \tilde{I} .

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- If (M, g, I) is (para-)Kähler, we call it a (para-)Kähler submanifold of \widetilde{M} .
- An almost (para-)Hermitian submanifold M together with a section \tilde{I} of $Q|_M$ is said to be totally (para-)complex if at each point $p \in M$, we have $LT_pM \perp T_pM$, for each $L \in Q_p$ with $\tilde{g}(L, \tilde{I}_p) = 0$.

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- Then the image G(M) is a real (2n-2)-dimensional submanifold of $G_{1,1}(\mathbb{C}_1^{n+1})$ such that,
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- G(M) is totally para-complex provided $|\mu| < 2$,
- There exists a section \tilde{I} of the bundle $Q|_{G(M)}$ satisfying $\tilde{I}^2 = 0$ and $\tilde{I}TG(M) = TG(M)$ provided $|\mu| = 2$.

Integral curves of ξ of Hopf hypersurfaces in \mathbb{CH}^n

 Relationship between the value of μ: Hopf curvature of Hopf hypersurface in CHⁿ and behavior of each integral curve γ of ξ is:

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- $|\mu| = 2 \Leftrightarrow \gamma$ is a horocycle in $\mathbb{C}\mathbb{H}^n$,
- $|\mu| > 2 \Leftrightarrow \gamma$ is a geodesic circle in \mathbb{CH}^n .