Hopf hypersurfaces in non-flat complex space forms

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- where J is the complex structure of $\widetilde{M}$.

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- the structure vector $\xi = -JN$ is an eigenvector of the shape operator A of M, i.e., $A\xi = \mu\xi$.
- It is known that when $c \neq 0$, the principal curvature μ (Hopf curvature) of ξ on Hopf hypersurface M in $\widetilde{M}(c)$ is constant (Y.Maeda, Ki-Suh).

Structure of Hopf hypersurfaces in $\mathbb{CP}^n(4)$

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- provided that the focal map $\phi_r : M \to \mathbb{CP}^n$, $\phi_r(x) = \exp_x(rN_x)$ $(x \in M$, $N_x \in T_x^{\perp}M$, $|N_x| = 1)$ has constant rank on M.

Hopf hypersurfaces in $\mathbb{CH}^n$ with $|\mu| > 2$

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- provided that the rank of the focal map is constant as Cecil-Ryan's Theorem.

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- a Hopf hypersurface with $|\mu| < 2$ in $\mathbb{CH}^n$ may be constructed from
- an arbitrary pair of Legendrian submanifolds in S^{2n-1}.

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- $M^{2n-1} \hookrightarrow \mathbb{CP}^n \leftrightsquigarrow \Sigma^{2n-2} \hookrightarrow G_2(\mathbb{C}^{n+1})$,

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- $M^{2n-1} \hookrightarrow \mathbb{CP}^n \ \leftrightsquigarrow \ \Sigma^{2n-2} \hookrightarrow G_2(\mathbb{C}^{n+1})$,
- $M^{2n-1} \hookrightarrow \mathbb{CH}^n \ \leftrightsquigarrow \ \Sigma^{2n-2} \hookrightarrow G_{1,1}(\mathbb{C}^{n+1}_1)$.

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- Then we have a Gauss map $G : M^{2n-1} \to G_2(\mathbb{C}^{n+1})$ of real hypersurface M in $\mathbb{C}P^n$.

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- For each $p \in \widetilde{M}$, there is a neighborhood U of p over which there exists a local frame field $\{\tilde{I}_1, \tilde{I}_2, \tilde{I}_3\}$ of Q satisfying

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- For any element $L \in Q_p$, $\tilde{g}_p$ is invariant by L, i.e., $\tilde{g}_p(LX, Y) + \tilde{g}_p(X, LY) = 0$ for $X, Y \in T_p\widetilde{M}$, $p \in \widetilde{M}$.

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- The vector bundle Q is parallel in $\mathrm{End}\, T\widetilde{M}$ with respect to the Riemannian connection $\widetilde{\nabla}$ associated with $\tilde{g}$.
- Complex 2-plane Grassmannian $G_2(\mathbb{C}^{n+1})$ is an example of quaternionic Kähler manifold.

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- (1) $\tilde{I}^2 = -1$, (2) $\tilde{I}TM = TM$ (cf. Alekseevsky-Marchiafava, 2001, Osaka J. Math.).
- We denote by I the almost complex structure on M induced from $\tilde{I}$.

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- If (M, g, I) is Kähler, we call it a Kähler submanifold of $\widetilde{M}$.
- An almost Hermitian submanifold M together with a section $\tilde{I}$ of $Q|_M$ is said to be totally complex if at each point $p \in M$, we have $LT_pM \perp T_pM$, for each $L \in Q_p$ with $\tilde{g}(L, \tilde{I}_p) = 0$.

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- An almost Hermitian submanifold M together with a section $\tilde{I}$ of $Q|_M$ is said to be totally complex if at each point $p \in M$, we have $LT_pM \perp T_pM$, for each $L \in Q_p$ with $\tilde{g}(L, \tilde{I}_p) = 0$.
- It is known that a $2m$ $(m \geq 2)$-dimensional almost Hermitian submanifold M^{2m} is Kähler if and only if it is totally complex.

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- Suppose M is a Hopf hypersurface.
- Then the image $G(M)$ is a real $(2n-2)$-dimensional totally complex submanifold of $G_2(\mathbb{C}^{n+1})$.

$\mathbb{CP}^1$-bundle over $G_2(\mathbb{C}^{n+1})$

- Let $\boldsymbol{E} = \{(\boldsymbol{\ell_1}, \boldsymbol{\ell_2}) |\ \boldsymbol{\ell_1}, \boldsymbol{\ell_2} \in \mathbb{CP}^{\boldsymbol{n}}, \boldsymbol{\ell_1} \perp \boldsymbol{\ell_2}\}$.

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- Let $E = \{(\ell_1, \ell_2) |\ \ell_1, \ell_2 \in \mathbb{CP}^n, \ell_1 \perp \ell_2\}$.
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- E is a total space of $\mathbb{CP}^1$-bundle over $G_2(\mathbb{C}^{n+1})$, with the projection $E \to G_2(\mathbb{C}^{n+1})$, $(\ell_1, \ell_2) \mapsto \ell_1 \oplus \ell_2$.

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- Each fiber of $E \to G_2(\mathbb{C}^{n+1})$ is identified with the base point of $G_2(\mathbb{C}^{n+1}) \cong \{\mathbb{CP}^1 \subset \mathbb{CP}^n\}$.

$\mathbb{C}P^1$-bundle over $G_2(\mathbb{C}^{n+1})$

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- Then $E = U(n+1)/U(1) \times U(1) \times U(n-1)$.
- E is a total space of $\mathbb{C}P^1$-bundle over $G_2(\mathbb{C}^{n+1})$, with the projection $E \to G_2(\mathbb{C}^{n+1})$, $(\ell_1, \ell_2) \mapsto \ell_1 \oplus \ell_2$.
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- We have another projection $\mathrm{pr}_1 : E \to \mathbb{C}P^n$, $\mathrm{pr}_1(\ell_1, \ell_2) = \ell_1$.

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- An almost complex structure $\tilde{I}$ in Q_p, where $p \in G_2(\mathbb{C}^{n+1}) \cong \{\mathbb{CP}^1 \subset \mathbb{CP}^n\}$.

Circle bundle over totally complex submanifold

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- We denote M_r $(0 \leq r < \pi/4)$ as a circle bundle over Σ such that each fiber of $M_r \to \Sigma$ is the equi-distance curve (circle) from the geodesic in $\mathbb{CP}^1 \cong \varphi(p)$ $(p \in \Sigma)$ corresponding to $\tilde{I} \in Q_{\varphi(p)}$.

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- where η denotes the bundle map.
- Hence $\Phi_r := \mathrm{pr}_1 \circ \eta : M_r \to \mathbb{CP}^n$ gives a real hypersurface provided that Φ_r is an immersion.

Frame field of totally complex submanifold

- For a totally complex immersion $\varphi : \Sigma^{n-1} \to G_2(\mathbb{C}^{n+1})$, we have a (local) frame field $g : \Sigma \to U(n+1)$, $g(p) = (u_1(p), u_2(p), \cdots, u_{n+1}(p))$ such that

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- We denote ω_{ij} $(1 \leq i, j \leq n+1)$ as components of the pull-back of Maurer-Cartan form $g^{-1}dg$ on $U(n+1)$.

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- suppose the map $\Phi_r : M_r \to \mathbb{CP}^n$ is an immersion, where $M_r \to \Sigma$ is a circle bundle as above.
- Then $\Phi_r(M_r)$ is a Hopf hypersurface in $\mathbb{CP}^n$ provided that $\mathrm{Im}\,\omega_{11} = \mathrm{Im}\,\omega_{22}$ and $\mathrm{Im}\,\omega_{12} = 0$ hold.

Example

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- $\pi(S^{2k+1} \times S^{2n-2k-1}) \subset \mathbb{C}P^n$ $(0 \leq k \leq n-1)$,
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- $\mathbb{C}P^k \times \mathbb{C}P^{n-k-1} \subset G_2(\mathbb{C}^{n+1})$.

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- H_1^{2n+1} is the principal fiber bundle over $\mathbb{CH}^n$ with the structure group S^1 and the fibration $\pi : H_1^{2n+1} \to \mathbb{CH}^n$.

Gauss map of real hypersurface in $\mathbb{CH}^n$

- Let $\Phi : M^{2n-1} \to \mathbb{CH}^n$ be an immersion and let N_p be a unit normal vector of M at $p \in M$.

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- Then we have a Gauss map $G : M^{2n-1} \to G_{1,1}(\mathbb{C}_1^{n+1})$ of real hypersurface M in $\mathbb{CH}^n$.

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Quaternions and quaternionic structure

- $\mathbb{H} = C(0,2)$, Quaternions: $q = q_0 + iq_1 + jq_2 + kq_3$,
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- $V = \{aI_1 + bI_2 + cI_3 \mid a, b, c \in \mathbb{R}\} \cong \mathfrak{su}(2) \cong \mathbb{R}^3$, and $Q = \{I \in V \mid I^2 = -1\} \cong S^2$.

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- `http://en.wikipedia.org/wiki/Split-quaternion`
- Introduced by James Cockle in 1849.

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- $\{I_1, I_2, I_3\}$, $I_1^2 = -1$, $I_2^2 = I_3^2 = 1$, $I_1 I_2 = -I_2 I_1 = -I_3$, $I_2 I_3 = -I_3 I_2 = I_1$, $I_3 I_1 = -I_1 I_3 = -I_2$ gives para-quaternionic structure,

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- $Q_+ = \{I \in \tilde{V} |\ I^2 = 1\} \cong S^2_1$: de-Sitter space, $Q_- = \{I \in \tilde{V} |\ I^2 = -1\} \cong H^2$: hyperbolic space, $Q_0 = \{I \in \tilde{V} |\ I^2 = 0\} \cong$ lightcone.

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- Complex $(1,1)$-plane Grassmannian $G_{1,1}(\mathbb{C}_1^{n+1})$ is an example of para-quaternionic Kähler manifold.

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- We denote by I the almost (para-)complex structure on M induced from $\tilde{I}$.

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- An almost (para-)Hermitian submanifold M together with a section $\tilde{I}$ of $Q|_M$ is said to be totally (para-)complex if at each point $p \in M$, we have $LT_pM \perp T_pM$, for each $L \in Q_p$ with $\tilde{g}(L, \tilde{I}_p) = 0$.

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- There exists a section $\tilde{I}$ of the bundle $Q|_{G(M)}$ satisfying $\tilde{I}^2 = 0$ and $\tilde{I}TG(M) = TG(M)$ provided $|\mu| = 2$.

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